The Proof of the Divergence of the Harmonic Series

Has the struggle for Christmas presents brought your bank account close to or, gods forbid, into the red? Although it is well beyond my competence to you help you more directly, I can at least give you a headstart on getting a \$1 million into your virtual pocket. However, as can be expected in a Maths Advent Calendar, the means will be strictly mathematical!

Assuming that I have by now got you on the edge of your seat, desperate to give Mathematics one more chance because of what it allegedly promises, I had better stop beating around the bush. If you have ever dealt with non-euclidean geometries, you will have, metaphorically, of course, encountered Bernhard Riemann, for it is him after whom Riemannian geometry is named. Although I do intend to devote some time to that field of mathematics someplace in the calendar, what we are interested in today is the Riemann zeta function. In case you have never heard about it, here it is in its full glory.

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, z \in \mathbb{C}$$

The Riemann zeta function determines the infinite series of the powers of the reciprocals of positive integers. The notion of this function has been around for centuries. In fact, it was in 1350 already that the French philosopher Nicole Oresme showed that for $z = 1$, in which case we call the series the harmonic series, the series diverges (Nahin 162). Does that seem somewhat counterintuitive? No worries, Oresme's proof is beautifully simple. ("Beautifully simple" would almost sound like a tautology in Mathematics.) Let us have a look at it.

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots =$$

$$= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$\zeta(1) > 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

In words, what Oresme does (or did, to be grammatically correct) is that he expands the series, groups the reciprocals in such a way that the last fraction in each pair of brackets is the reciprocal of a power of 2, and replaces each preceding term within those brackets with a number slightly smaller: the very reciprocal that can be found at the end of the group. By this, he obtains a new series whose sum is apparently less than $\zeta(1)$. However, this series, being the sum of an infinite number of $\frac{1}{2}$s, diverges, and so must $\zeta(1)$.

Sadly, the time has now come for me to confess that the rest of the job will not be such a piece of cake. After Oresme's discovery, it took another few hundred years before Leonhard Euler (*the* Euler) solved $\zeta(2)$, after the unsuccess of such hegemons as John Bernoulli, Wilhelm Leibniz, and John Pell, through clever use of the sine. We have only proceeded from $z = 1$ to $z = 2$, and look how much more complicated matters have got.

Where is that \$1 million, you are asking; I can sense your growing feeling of suspicion. We are getting there.

After successfully determining the value of the function for a few concrete input values, one talented man, as mathematicians sometimes tend to, decided to make a conjecture of an altogether different dimension than what had previously been explored. This talented man was Bernhard Riemann, and the hypothesis he brought into the mathematical word was that all of the nontrivial complex zeros of the zeta function, that is to say, those zeros which are not negative even integers, are on the critical line $x = \frac{1}{2}$ ('Riemann Zeta Function').

So far, nobody has managed to either prove or disprove the hypothesis. So significant has the problem been deemed that it has found its way onto the list of The Millennium Prize Problems established by The Clay Mathematics Institute ('Riemann Hypothesis'). I presume that it will have dawned on you by now. Yes, the sum allocated to the solution of each of these problems is precisely $1 million.

I hope that I have not left you with the feeling of anger over my having inconsiderately given you false hope, that was certainly not my intention. The purpose of this article was merely to have you comprehend $\zeta(1)$ and to have you smile over understanding just a snippet of Maths. If I have managed to do that, I will consider my task as accomplished.

… but what if one of those problems is waiting for you to solve it? No need to rush. It does not necessarily have to be today or tomorrow, you have years and years ahead of you to ponder it. Also, if the Riemann Hypothesis is not your cup of tea, there are another five Millennium Prize Problems to choose from…

Sources

Nahin, Paul. *An Imaginary Tale: The Story of [the Square Root of Minus One]*. Princeton University Press, 1998.

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