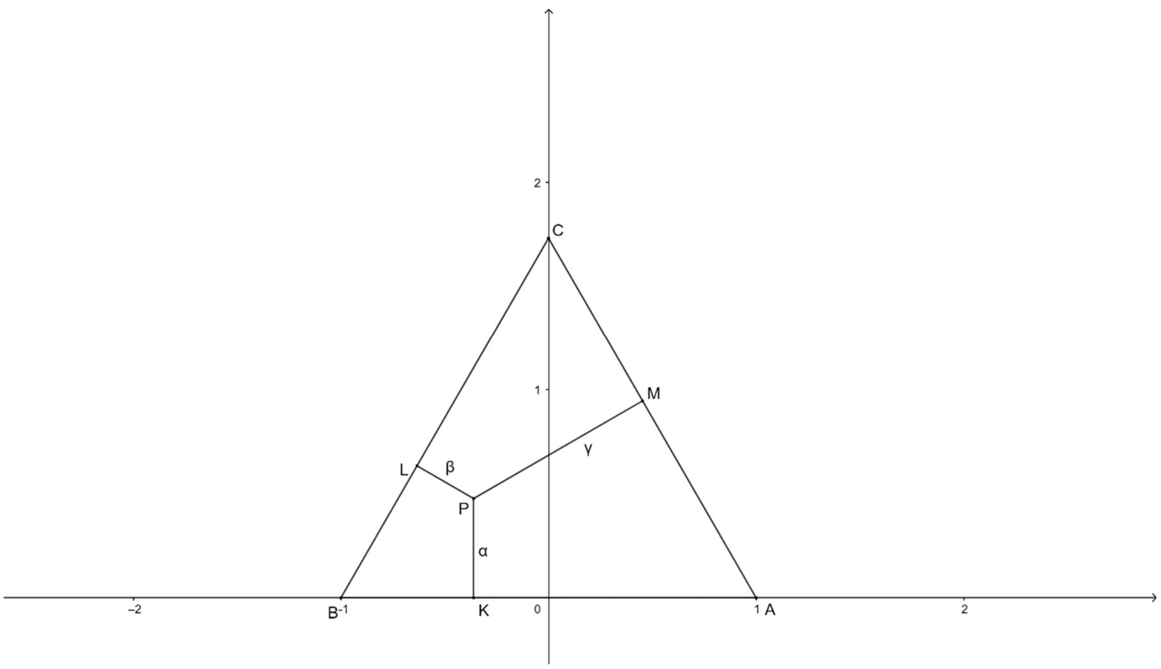
The Spaghetti Problem

We have all been there. At the time you opened that massive packet of spaghetti and only used a fraction of its contents, you put the bag with the rest of the pasta back into the cupboard without taking any precautionary measures whatsoever. The next time you arrive in the kitchen with pasta on your mind, you forget that the packet is already opened, and a disaster is born. You stretch out to grasp the thing, manhandle it, and down to the floor fly myriad barley sticks. Some of them break into two pieces, some into smithereens, and some, a precious few, happen to split into three parts exactly. When experiencing your own version of the disaster, did you even in the shortest of moments consider what the probability was of the constructability of a triangle from the three pieces of that one pasta that broke into three pieces? Perhaps not. No worries, other mathematicians have in a problem called the Spaghetti Problem. Today, we will define this problem, concoct a useful probabilistic model for it, and solve it!

> **The Spaghetti Problem.** *A spaghetti stick, dropped on the floor, breaks at random into three pieces. What is the probability that the three parts obtained are the sides of a triangle?* (Ionascu and Prajitura 3)

Now that we have put our problem into an individual half boldened, half italicised paragraph, we may move on to the probabilistic model itself! The one we will be using is the one first introduced by Poincaré in a series of lectures at Sorbonne University and later published in his textbook *Calcul des Probabilités*. (Almost a century later, the model was also adopted in the article thanks to which I now have a fairly functioning IB Extended Essay topic (Ionascu and Prajitura).) I include a GeoGebra-generated sketch of it below.



We may assume, without losing generality (now, that would be a loss worth mourning in Mathematics!), that the stick in question has a length of $l = \sqrt{3}$. Its three parts, α, β, and γ can be obtained in the following fashion. Let ABC be an equilateral triangle with sides of length $a = 2$, the vertices of which have the coordinates $A(1, 0)$, $B(-1, 0)$, and $C(0, \sqrt{3})$ in the Cartesian coordinate system. If we choose a random point P inside triangle ABC, the three parts of the stick can be defined as the distances of this point P to the sides of the triangle ABC. In symbols (a mathsy version of "in other words”), $\alpha = |PK|$, $\beta = |PL|$, $\gamma = |PM|$. That $\alpha + \beta + \gamma = \sqrt{3}$ can be shown by proving Viviani’s Theorem. Never heard of it? Nor had I until few months ago. Let us have a look at it.

Viviani’s Theorem states that for an equilateral triangle, the sum of the altitudes from any point in the triangle is equal to the altitude from a vertex of the triangle to the other side (‘Viviani’s Theorem’). The proof thereof? By choosing point P, we can divide the equilateral triangle ABC into three separate triangles, BAP, BCP, and CAP, whose heights are α, β, and γ, and the sum of whose areas is equal to the area of the triangle ABC. It follows that $\frac{a\alpha}{2} + \frac{a\beta}{2} + \frac{a\gamma}{2} = a\frac{\sqrt{3}}{2}a\frac{1}{2}$ or $(\alpha + \beta + \gamma) = \frac{a}{2}\sqrt{3}$, and $\frac{a}{2}\sqrt{3}$ is, indeed, the altitude of the triangle!

For $a = 2$, we have $\alpha + \beta + \gamma = \sqrt{3}$. The probability of event E is then equal to the proportion of the area within the triangle representing E to the area of the entire triangle. In our case, E is the set of all the points P which give such α, β, and γ that we can make a unique triangle out of them. Have I confused you slightly? Are you scratching your head in wonder (I would seriously like to know why people do that when in doubt, but this pondering of mine is hardly what this article is meant to cover), ruminating about how many marbles I have managed to lose during my short life? Stop scratching and have more faith in your mathematical abilities. Look, what is it that we want from α, β, and γ? Which inequality (wink, wink) do these values have to satisfy so that we can make a triangle out of them?

You are absolutely right, the triangle inequality! For a triangle to exist, $\max(\alpha, \beta, \gamma) < \frac{\alpha+\beta+\gamma}{2}$, where $\frac{\alpha+\beta+\gamma}{2} = \frac{\sqrt{3}}{2}$, as we are breaking a spaghetti of length $l = \sqrt{3}$. In our triangle ABC, this inequality is represented by the region given by the midpoints of the sides AB, BC, CA. The area of this region is $\frac{1}{4}$ of that of the triangle, so the probability of the constructability of a triangle with sides α, β, and γ is $\frac{1}{4}$.

I have no clue whatsoever how this can turn out to be useful in everyday life of either you or me. I confess that whole-length spaghetti is much fancier than ten-centimetre noodles. (My favourite birthday dinner used to be spaghetti bolognese, and the most delightful version I was ever served was that with metre-long spaghetti.) However, I can confidently tell you that I have tired of proving to people that the Mathematics I love can, in some way or other, be *applied*. It is just that Mathematics of this kind makes me smile. Does it make you smile, too? Whatever the impression this article has had on you, I hope that it has left you with one thing at least: You will never see spaghetti in the same light again.

Sources

Ionascu, Eugen, and Gabriel Prajitura. ‘Things to Do with a Broken Stick’. ArXiv.Org, 5 Sept. 2010, arxiv.org/pdf/1009.0890.pdf.

Poincaré, Henri. Calcul Des Probabilités. 2nd ed., Paris, Jacques Gabay, 1912.

‘Viviani’s Theorem’. Art of Problem Solving, © 2020 AoPS Incorporated, artofproblemsolving.com/wiki/index.php/Viviani%27s_theorem. Accessed 25 Oct. 2020.