AM-GM Inequality: Part 2

In the first part of our adventurous journey through the lands of handy inequalities, we not only listened to my entertaining ramble about my discontent at not understanding things, but we also made clear the difference between sets and multisets, and we defined and proved the Rearrangement Inequality. If you have not read Part 1 (window 21), but neither of the latter two topics rings a bell, I would highly recommend that you browse the calendar and acquaint yourself with the notions. If in a rush, I publicly declare that you may skip the first paragraph without my getting gravely offended.

Given a multiset $A = \{a_1, a_2, \dots, a_n\}$ of positive real numbers, we define the arithmetic mean of the multiset as $A_n = \frac{a_1+a_2+\cdots+a_n}{n}$ and the geometric mean of the multiset as $G_n = \sqrt[n]{a_1 a_2 \dots a_n}$. The *unweighted* Arithmetic Mean-Geometric Mean (or AM-GM) Inequality states that $\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}$ or, using summation and product symbols, $\sum_{i=1}^{n} \frac{a_i}{n} \geq \prod_{i=1}^{n} a_i^{\frac{1}{n}}$.

Now, there are several ways of proving the AM-GM Inequality, out of which the most interesting I find the proof by Cauchy Induction. (You may stay in the dark regarding the details of that term. However, Cauchy Induction happens to be a rather handsome type of induction, so do not hesitate to take up the initiative and investigate!)

We will begin by showing that the inequality holds for two numbers, let us call them a and b. First note that $\{\sqrt{a}, \sqrt{b}\}$ and $\{\sqrt{a}, \sqrt{b}\}$ are similarly sorted permutations of the multiset $\{\sqrt{a}, \sqrt{b}\}$. We can express the product of their square roots as $\sqrt{ab} = \frac{\sqrt{a}\sqrt{b}+\sqrt{b}\sqrt{a}}{2}$. The Rearrangement Inequality states that $\frac{\sqrt{a}\sqrt{b}+\sqrt{b}\sqrt{a}}{2} \leq \frac{\sqrt{a}\sqrt{a}+\sqrt{b}\sqrt{b}}{2} = \frac{a+b}{2}$, from which we may conclude $\sqrt{ab} \leq \frac{a+b}{2}$, with equality occurring when a and b are equal, which is a statement no other than that of the AM-GM Inequality for two numbers!

The next step on the path towards proving the general case is to show that if the inequality holds for n variables, it also holds for $2n$ variables. Consider the multiset permutation $\{a_1, a_2, \dots, a_{2n}\}$. We define $A_n = \frac{a_1+a_2+\cdots+a_n}{n}$, $A'_n = \frac{a_{n+1}+a_{n+2}+\cdots+a_{2n}}{n}$, $A_{2n} = \frac{a_1+a_2+\cdots+a_{2n}}{2n}$, $G_n = \sqrt[n]{a_1 a_2 \dots a_n}$, and $G'_n = \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}$. Our initial assumption can be expressed symbolically as $G_n \leq A_n$ and $G'_n \leq A'_n$, and, the sizes of A_n and A'_n being equal, we know that $A_{2n} = \frac{A_n + A'_n}{2}$. What do we know about G_{2n}? From the definition of the geometric mean, we have $G_{2n} = \sqrt[2n]{a_1 a_2 \dots a_{2n}} = \sqrt{\sqrt[n]{a_1 a_2 \dots a_{2n}}} = \sqrt{\sqrt[n]{a_1 a_2 \dots a_n}\sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}} = \sqrt{G_n G'_n}$. The AM-GM Inequality for two variables gives $\sqrt{G_n G'_n} \leq \frac{G_n + G'_n}{2} \leq \frac{A_n + A'_n}{2} = A_{2n}$. In case you do not see it, we have made the discovery of $G_{2n} \leq A_{2n}$, as desired! In other words, knowing that the AM-GM Inequality holds for two variables, it must also hold for all those cases in which we have 2^k variables for a natural k. To put it another way, for each n, there exists an N such that the AM-GM Inequality holds for N variables.

The next step is to show that if the inequality holds for N variables and $N > n$, it also holds for n variables, whose arithmetic mean is A_n and whose geometric mean is G_n. We will do so by creating a clever "artificial" set B of N numbers, where we set $b_k = a_k$ for $1 \leq k \leq n$ and $b_k = A_n$ for $n \leq k \leq N$. How do we know that this is the right approach? Gape in awe. Applying basic rules of algebra and just a spark or two of mathsy magic, we can confidently establish that $\sqrt[N]{G_n^n} \cdot \sqrt[N]{A_n^{N-n}} =$

$\sqrt[N]{a_1 a_2 \dots a_n \cdot A_n^{N-n}} = \prod_{i=1}^{N} b_i^{\frac{1}{N}} \leq \sum_{i=1}^{N} \frac{b_i}{N} = A_N = A_n$. In short, $\sqrt[N]{G_n^n} \cdot \sqrt[N]{A_n^{N-n}} \leq A_n$ and, therefore $\sqrt[N]{G_n^n} \leq \frac{A_n}{\sqrt[N]{A_n^{N-n}}} = \sqrt[N]{\frac{A_n^N}{A_n^{N-n}}} = \sqrt[N]{A_n^n}$. Raising both sides of the last inequality to the power of N and subsequently the power of $\frac{1}{n}$, we obtain $G_n \leq A_n$. Again, the inequality becomes an equality exactly when all as are equal.

Now, you may have been wondering what it was precisely that I meant by the term “unweighted”. In that case, please accept my congratulations on magnificent literacy skills. (Let us be honest with ourselves. How many of us skip the words that do not carry the *main* meaning in a sentence?) It may surprise you that we have, in fact, been examining only a special case of the general AM-GM Inequality, for we have been assigning equal weight to each of the variables in a set of size n, namely the weight $\frac{1}{n}$. In the general case, we deal not only with the n positive real numbers $a_1, a_2, \dots, a_n$ but also with another n positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ which have the additional characteristic of $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. It can be proven that $\sum_{i=1}^{n} \lambda_i a_i \geq \prod_{i=1}^{n} a_i^{\lambda_i}$. However, the proof of that involves, from what little I have seen, a not-insignificant amount of fairly advanced calculus, which, if you do not mind, I would like to save for another day.

Another thing that might have been jumping up and down restlessly at the back of your mind is the question of the usefulness of such an inequality. On other occasions, I have given you the answer: It’s cute! That’s why! This, of course, remains to be true, especially because such an inequality would not appear as obvious at first sight. However, there is one more factor that some of you might appreciate in the future. The creators of advanced olympiad problems seem to have a predilection for this little hack, so you might as well be aware of its existence if you want to be taking the path of a competitor seriously… If not, no worries. Making friends with Mathematics is possible even without the accolades!

Sources

‘Proofs of AM-GM’. Art of Problem Solving, artofproblemsolving.com/wiki/index.php/Proofs_of_AM-GM. Accessed 22 Nov. 2020.