The Radical Conjugate Root Theorem

You might have heard of the famous Complex Conjugate Root Theorem which states that if $p(x) = a_0x^0 + a_1x^1 + a_2x^2 + \ldots + a_nx^n$ is a polynomial with real coefficients a_0, a_1, …, a_n and $a + bi$ is a root of the equation $p(x) = 0$, where $i = \sqrt{-1}$, $a - bi$ is also a root ('Conjugate Root Theorem'). However, does the term "the radical conjugate theorem" ring a similar bell? If not, no worries. Today, I shall introduce it to you and we will prove it together. No point in beating around the bush; we may radically begin!

We must first define this novel conjugate. The radical conjugate of an irrational number $a + b\sqrt{c}$, where $a, b, c \in \mathbb{Q}$, is $\overline{a + b\sqrt{c}} = a - b\sqrt{c}$. Since it will prove incredibly useful later on, we will quickly deduce one peculiar property that performing basic algebra on our conjugate has. Let us claim that, given the radical numbers $a + b\sqrt{c}$ and $d + e\sqrt{c}$, where $a, b, c, d, e \in \mathbb{Q}$, it holds that $\overline{a + b\sqrt{c}} \cdot \overline{d + e\sqrt{c}} = \overline{(a + b\sqrt{c}) \cdot (d + e\sqrt{c})}$. Thankfully, this statement can be proven directly. Watch in awe.

$$\overline{a + b\sqrt{c}} \cdot \overline{d + e\sqrt{c}} = (a - b\sqrt{c}) \cdot (d - e\sqrt{c}) =$$

$$= (ad + bec) - (ae\sqrt{c} + bd\sqrt{c}) =$$

$$= \overline{(ad + bec) + (ae\sqrt{c} + bd\sqrt{c})} = \overline{(a + b\sqrt{c}) \cdot (d + e\sqrt{c})}$$

Believe it or not, these are all the tools we need today. (I may or may not be hinting at tougher times yet to come!) Hence, we may now proceed to the almighty theorem itself. The Radical Conjugate Theorem states that if $p(x) = a_0x^0 + a_1x^1 + a_2x^2 + \ldots + a_nx^n$ is a polynomial with rational coefficients a_0, a_1, …, a_n, and $a + b\sqrt{c}$ is a root of the equation $p(x) = 0$, where $a, b, c \in \mathbb{Q}$, then $a - b\sqrt{c}$ is also a root of $p(x)$. Does that sound rather far-fetched? Worry not and follow me.

We will first show that $\overline{(a + b\sqrt{c})^n} = (\overline{a + b\sqrt{c}})^n$.

$$\overline{(a + b\sqrt{c})^n} = \overline{a^n + \binom{n}{1}a^{n-1}b\sqrt{c} + \binom{n}{2}a^{n-2}(b\sqrt{c})^2 + \binom{n}{3}a^{n-3}(b\sqrt{c})^3 + \ldots + (b\sqrt{c})^n} =$$

$$= \overline{a^n + \binom{n}{2}a^{n-2}c \cdot b^2 + \ldots + \binom{n}{1}a^{n-1}b\sqrt{c} + \binom{n}{3}a^{n-3}c \cdot b^3\sqrt{c} + \ldots} =$$

$$= \overline{a^n + \binom{n}{2}a^{n-2}c \cdot b^2 + \cdots + \left(\binom{n}{1}a^{n-1}b + \binom{n}{3}a^{n-3}2 \cdot b^3 + \cdots\right)\sqrt{c}} =$$

$$= a^n + \binom{n}{2}a^{n-2}c \cdot b^2 + \ldots - \left(\binom{n}{1}a^{n-1}b + \binom{n}{3}a^{n-3}2 \cdot b^3 + \ldots\right)\sqrt{c} =$$

$$= a^n + \binom{n}{2}a^{n-2}c \cdot b^2 + \ldots - \binom{n}{1}a^{n-1}b\sqrt{c} - \binom{n}{3}a^{n-3}2 \cdot b^3\sqrt{c} - \ldots =$$

$$= a^n - \binom{n}{1}a^{n-1}b\sqrt{c} + \binom{n}{2}a^{n-2}(b\sqrt{c})^2 - \binom{n}{3}a^{n-3}(b\sqrt{c})^3 + \ldots =$$

$$= a^n + \binom{n}{1}a^{n-1}(-b\sqrt{c}) + \binom{n}{2}a^{n-2}(-b\sqrt{c})^2 + \binom{n}{3}a^{n-3}(-b\sqrt{c})^3 + \ldots + (-b\sqrt{c})^n =$$

$$= (a - b\sqrt{c})^n = (\overline{a + b\sqrt{c}})^n$$

Now, we already know that $p\left(a+b\sqrt{c}\right)=0$. Let us see what happens when we plug $\overline{a+b\sqrt{c}} = a-b\sqrt{c}$ into the polynomial function!

$$p(\overline{a+b\sqrt{c}}) = a_0(\overline{a+b\sqrt{c}})^0 + a_1(\overline{a+b\sqrt{c}})^1 + a_2(\overline{a+b\sqrt{c}})^2 + \ldots + a_n(\overline{a+b\sqrt{c}})^n =$$

$$= a_0\overline{(a+b\sqrt{c})^0} + a_1\overline{(a+b\sqrt{c})^1} + a_2\overline{(a+b\sqrt{c})^2} + \ldots + a_n\overline{(a+b\sqrt{c})^n} =$$

$$= \overline{a_0(a+b\sqrt{c})^0} + \overline{a_1(a+b\sqrt{c})^1} + \overline{a_2(a+b\sqrt{c})^2} + \ldots + \overline{a_n(a+b\sqrt{c})^n} =$$

$$= \overline{a_0(a+b\sqrt{c})^0 + a_1(a+b\sqrt{c})^1 + a_2(a+b\sqrt{c})^2 + \ldots + a_n(a+b\sqrt{c})^n} =$$

$$= \overline{p(a+b\sqrt{c})} = \overline{0} = 0$$

We have just shown that $p(\overline{a+b\sqrt{c}})=0$. In other words, our radical conjugate $a-b\sqrt{c}$ is a root of $p(x)=0$! (That last exclamation mark is genuinely just an exclamation mark, not a factorial of zero. That would wreak *some* havoc!) However, we are not quite finished yet. Before bidding you a fond farewell, I would like to draw your attention to one assumption we made in the process of proving our theorem, one you might have noticed and pondered its legitimacy. Without explicitly stating so, I have presented it as a fact that $a_n\overline{(a+b\sqrt{c})^n} = \overline{a_n(a+b\sqrt{c})^n}$. This is only possible since a_n, having no radical part, is its own radical conjugate $a_n = a_n + 0\sqrt{c} = \overline{a_n - 0\sqrt{c}} = \overline{a_n}$, a fact certified by our tool. A fairytale ending to a dramatic story? I should say so!

Sources

‘Conjugate Root Theorem’. *Art of Problem Solving*, © 2020 AoPS Incorporated, artofproblemsolving.com/wiki/index.php/Conjugate_Root_Theorem. Accessed 2 Nov. 2020.