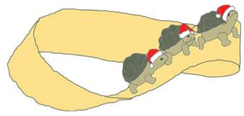

The Stable Marriage Problem

When ambitiously applying to my maximum of five British universities, I twice stumbled upon the peculiar phenomenon of a single seat of learning split into multiple colleges, between which I was to state my humble preferences. Learning that my application would be considered by each college individually, I was intellectually intrigued. If those two universities wanted to account for my college preferences, as well as for the colleges' preferences between prospective applicants, did they have an algorithm for pairing applicants and colleges so that everybody would be as happy as possible?

Luckily enough, I got an answer when listening to a podcast in the series *Secrets of Mathematics* (sadly, I no longer remember which episode it was), where the topic of college-applicant pairing was being discussed. The algorithm I had assumed existed, and its contemporary form was based on a problem concocted fifty years before, the Stable Marriage Problem. As I found the information in the podcast rather insufficient, I decided to read the paper which had introduced the problem to the mathematical community (Gale and Shapley).

The primordial version of the Stable Marriage Problem stated therein is the following.

> *In an instance of the Stable Marriage Problem, each of* n *men and* n *women lists the members of the opposite sex in order of preference. A stable marriage or matching is defined as a complete matching of men and women with the property that there are no two couples* (m, w) *and* (m', w') *such that* m *prefers* w' *to* w *and* w' *prefers* m *to* m'. (Irving and Gusfield, 532)

You might argue that this problem has very little in common with that of the college-applicant pairing. Now, I would like to point out that the only difference is that with the college-applicant pairing, the colleges are seeking a number of applicants greater than one, whereas, in the Stable Marriage Problem, each man is to be paired with one woman only and *vice versa*. Otherwise, the characteristics of the problems, such as there being two separate groups and a set of preference lists, are the same. I am getting too serious. Forget about the university applications (daunting a process, indeed), back to the Stable Marriage Problem!

As given in the statement of the problem, each of the n men has a list of the n women in order of his preference, and each of the n women has a list of the n men in order of her preference. In other words, the person in position j of i's list is person i's jth choice. Below is an example of these preference lists for $n = 8$.

0:	2	0	4	6	3	1	7	5
1:	5	0	2	3	7	6	4	1
2:	6	3	2	5	4	0	1	7
3:	4	2	7	1	5	0	3	6
4:	3	0	1	7	6	2	5	4
5:	5	1	4	6	7	3	2	0
6:	6	7	0	5	1	2	3	4
7:	1	5	6	0	7	2	3	4

male preference lists

0:	3	2	7	0	1	4	6	5
1:	2	6	4	7	5	3	0	1
2:	6	4	7	2	5	1	0	3
3:	5	3	1	6	2	0	4	7
4:	7	6	0	4	5	3	2	1
5:	4	3	6	5	1	7	2	0
6:	0	3	4	5	1	7	2	6
7:	1	4	3	2	6	7	0	5

female preference lists

It was shown by Gale and Shapley already, the authors of the paper I mentioned, that there is at least one stable matching for each instance of the problem. They did so nicely and straightforwardly by providing an algorithm which generates this matching. Clever men!

What is this algorithm? Brace yourself and, if needed, read this paragraph several times. I sincerely promise that you will get the hang of it. The Gale–Shapley algorithm, for that it what it is somewhat uninspiringly called, consists of a sequence of proposals from men to women. In the first round of proposals, each of the n men proposes to the woman on top of his list. Afterwards, each of the n women rejects all but her favourite from those men who have proposed to her and puts her favourite on hold. In the second round, all of the previously rejected men propose to their second choices. Each woman receiving proposals then puts on hold her favourite among the group consisting of her old proposer and her new proposers and rejects the rest. Similarly, the third, fourth, etc., rounds are carried out, until each woman has received a proposal. That such a situation has to come about is given by the fact that as long as any woman has not been proposed to, the number of men put on hold in a round is less than n and the number of men rejected in a round is greater than zero, so there will be new proposals in the next round. At the same time, no man can propose to the same woman more than once. Eventually, every woman will have received a proposal, for the process is finite! (I include below a computer-generated sequence of proposals for the concrete example introduced earlier. Feel free to go through it!)

Round 1

m_0 proposes to w_2

m_1 proposes to w_5

m_2 proposes to w_6

m_3 proposes to w_4

m_4 proposes to w_3

m_5 proposes to w_5

m_6 proposes to w_6

m_7 proposes to w_1

m_7 becomes paired with w_1

m_0 becomes paired with w_2

m_4 becomes paired with w_3

m_3 becomes paired with w_4

w_5 ranks m_1 as 4

w_5 ranks m_5 as 3

m_5 becomes paired with w_5

w_6 ranks m_2 as 6

w_6 ranks m_6 as 7

m_2 becomes paired with w_6

m_1, m_6, w_0, w_7 remain single

Round 2

m_1 proposes to w_0

m_6 proposes to w_7

m_1 becomes paired with w_0

m_6 becomes paired with w_7

However, is the resulting matching $S = \{(m_0, w_0), \dots, (m_{n-1}, w_{n-1})\}$ truly stable? It is! Suppose m_i prefers w_j $(0 \leq i, j \leq n-1)$ to w_i. Bearing in mind the characteristics of the procedure described above, we know that m_i must have proposed to w_j at some stage of the proposals and subsequently been rejected in favour of somebody w_j liked better than him. Thus, w_j prefers m_j, her partner in S, to m_i, and there is no instability.

So what? you might ask. How does this all affect you? Well, you might finally feel justified in finding that finding the right partner in a world of a few billion people is tough. Or, like me, you can have just that little more faith in the seemingly random college system some universities have adopted and stubbornly refuse to let go of. Or, and this would be the best possible impact my article could have, you might admit that there is sometimes more to Mathematics than you once thought there was.

(However, no matter how enthusiastic you get, I would generally advise against writing your IB Internal Assessment in Mathematics on “efficient algorithms” for finding the “optimal solution” to the Stable Marriage Problem. That is a topic that turns out to be just a tad more complicated than I would like it to be.)

Sources

Gale, David, and Lloyd S. Shapley. ‘College Admissions and the Stability of Marriage’. The American Mathematical Monthly, vol. 120, no. 5, 2013, p. 386. Crossref, doi:10.4169/amer.math.monthly.120.05.386.

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