The Cubic: Part 2

Welcome back to our ultimate expedition into the murky but magical waters of solving the cubic! Last time, we devoted a not-insignificant amount of attention to the depressed cubic; hence, if you have not read Part 1 (window 9), I would highly recommend that you do so, for implications will be drawn today that depend on what was being discussed there. If time is scarce, you are relieved of the duty of reading the first paragraph. It is merely one of my rants about my failed life of a mathematician. However, the rest is quite important. (Off you go! Yes, you!)

In Part 1, I claimed that the transition from the depressed to the general cubic was simple. My wording was not very accurate, though, for we will be solving the general cubic by moulding it back into the depressed form! Consider the general cubic $x^3 + ax^2 + bx + c = 0$, where a, b, and c are all real non-zero. (In case any of them is zero, there are simpler ways of solving the equation, and our magic is a tad out of place.) We can then expand on our substitution by introducing yet another variable y such that $x = y - \frac{1}{3}a$. Marvel at the simplicity of that for a short while, please.

Done with the marvelling? Good. Let us rewrite our cubic equation in terms of the newcomer, $\left(y - \frac{1}{3}a\right)^3 + a\left(y - \frac{1}{3}a\right)^2 + b\left(y - \frac{1}{3}a\right) + c = 0$. Opening the brackets, we get $y^3 - y^2a + \frac{1}{3}ya^2 - \frac{1}{27}a^3 + y^2a - \frac{2}{3}ya^2 + \frac{1}{9}a^3 + yb - \frac{1}{3}ab + c = 0$, and, after gathering relevant terms, we are left with $y^3 + \left(b - \frac{1}{3}a^2\right)y = -\frac{2}{27}a^3 + \frac{1}{3}ab - c$. Now, if this creation of Gerolamo Cardano does not deserve a standing ovation, I sincerely do not know what does, for we have just rewritten our general cubic as a depressed cubic $x^3 + px = q$, where $p = b - \frac{1}{3}a^2$ and $q = -\frac{2}{27}a^3 + \frac{1}{3}ab - c$. We already know how to face that issue. Therefore, we are done with the general case, too!

Well, not really. There is still that teensy catch I mentioned in Part 1 and which has not been magicked away yet, significant quanta of magic though we have been using throughout. Let us, instead of $x^3 + px = q$, consider $x^3 = px + q$ and substitute $-p$ into our former expression for x instead of p. We then get $x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$. As long as $\frac{q^2}{4} > \frac{p^3}{27}$, all goes well; the moment $\frac{q^2}{4} < \frac{p^3}{27}$, we have an expression for x which requires the calculation of the cube root of imaginary numbers. Now, if you have encountered imaginary numbers along your mathematical path, you will know that answering the question of the cube root of an imaginary number is ambiguous. Were we dealing with one such root only, we would find a legitimate root of our cubic equation. However, what guarantees that from all the combinations of the cube roots available, we have chosen the right ones? The truth is, I do not know. I have tried asking the Internet but have failed. All the suggestions I received appeared somewhat off, for none of them subtracted from the ambiguousness of what I was already facing. Instead, they all made me even more confused. I am disappointed in myself that I am unable to provide you with an ultimate answer on that account, but I solemnly swear that I have not closed the case of the general cubic just yet.

However, the promise I made in Part 1 was not entirely in vain, for I am still able to offer to you a solution to those depressed cubics for which $\frac{q^2}{4} < \frac{p^3}{27}$. It is only that we will have to bid a fond farewell to Cardano and, instead, pay a visit to François Viète, who conceived an ingenious solution elegantly skirting around the stumbling block we (I) have encountered. In Part 3, we will use secondary-school-level knowledge of complex numbers to prove one trigonometric identity and then bring our journey to a successful end. Until then, goodbye!

Sources

Nahin, Paul. *An Imaginary Tale: The Story of [the Square Root of Minus One]*. Princeton University Press, 1998.