Cauchy-Schwarz Inequality

"Oh, Klára," I can hear you moan, "yet another inequality? Yet another proof? You have already inundated me with such a barrage of both that I will be dreaming of less than or equal to signs for the rest of my life!" (The truth to be told, I do not yet know how I will be ordering my articles. If I decide to put Cauchy and Schwarz somewhere at the beginning of December, your moaning is now more along the lines of, "Oh, Klára, yet another inequality? Yet another proof? You will already be inundating me…" If that is the case, please accept my sincere congratulations on your remarkable future-reading skills!) The thing is that all of these inequalities, their proofs including, are just so, so pretty. You start off by getting something that appears as completely counterintuitive or that simply would never have crossed your mind, and you end up not only a little more mathematically knowledgeable but also satisfied by the fact that you understand the knowledge you have just gained.

Enough of my case for inequalities and their proofs, Messrs. Cauchy and Schwarz, the independent founders of the inequality under not-so-intense scrutiny today, are waiting for their creation to be presented to you. The Cauchy-Schwarz Inequality states that for any real numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$, it holds that $(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$, with equality in those cases only where there exists a real number μ such that for all i, $1 \leq i \leq n$, $\mu a_i = b_i$. Presented so, the inequality does seem somewhat arbitrary, indeed. However, it only suffices to translate the statement into vector form, and all becomes clearer! Instead of just focusing on the real numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$, let us consider the vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$, and let us call the angle between these two vectors θ. We can translate our knowledge on vectors back to the naked real numbers as $\|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2(\theta) = (\vec{a} \cdot \vec{b})^2 = (\sum_{i=1}^{n} a_i b_i)^2$ and $\|\vec{a}\|^2 \|\vec{b}\|^2 = (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$. Since $0 \leq \cos^2(\theta) \leq 1$ (for the angle between two vectors is always the one for which $0 \leq \theta \leq \pi$), we can express the relationship between the two values as either $\|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2(\theta) \leq \|\vec{a}\|^2 \|\vec{b}\|^2$ or $(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$, which is none other than the Cauchy-Schwarz Inequality!

Now, as is often the catch when dealing with a mathematical concept coming across as so simple, we have only been considering the elementary form of the inequality. There is also a complex form and a general form. Regarding the former, I have not found a clear proof designated for it, but I have the suspicion that the proof of the latter might apply. As far as the latter is concerned, the proof entails fairly advanced vector algebra, the kind I would first have to read upon and subsequently introduce to you to be able to look at the proof itself! However, in delving so deep (much as I am intrigued!), we would be losing some of the breadth I am aiming at with the MAC. However, if any of you was so interested as to be willing to cast all shyness aside and contact me, I would be more than happy to throw myself into the exploration of the complex and general. Let me know.

Sources

'Cauchy-Schwarz Inequality'. *Art of Problem Solving*, artofproblemsolving.com/wiki/index.php/Cauchy-Schwarz_Inequality. Accessed 25 Nov. 2020.