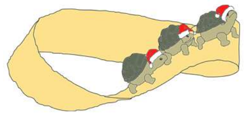

The Cubic: Part 1

I recall the first time I encountered the solution to the general cubic equation. It was Spring 2018, and I could be found on a week-long selective Maths camp. Throughout this camp, made up of lectures on Mathematics and a multitude of mathematically oriented games, I felt that a mistake must have occurred in the admissions process, for I seemed to be much less intellectually well-off than my peers, most of whom were younger than me. However, my conviction was undermined somewhat when I realised, during a lecture, that what had appeared as intellectual superiority was, to an extent, a matter of confidence. Although the mathematical aptitude of the individual participants varied, we were no more than a group of amateur enthusiasts, not a group consisting of one amateur enthusiast (me) and an overwhelming majority of mathematical gods. The lecture in question, at whose end nobody in the room doubted the existence of black magic, was on the solution to the general cubic.

By now, I assume, my motivation must be clear. Two years ago, I experienced genuine awe at watching live the unfolding of something that brilliant mathematical minds had introduced to the community two hundred years before. This December, I would like to transfer at least some of this awe to you. It is necessary that we divide the article into three parts, for to cover the topic in the manner it deserves, we will have to go over more ground than would fit into one piece of text. Let us get down to work!

In algebra, the cubic is a one-variable equation of the form $a_3x^3 + a_2x^2 + a_1x^1 + a_0x^0 = 0$, where a_3 is non-zero. To simplify matters, we will divide the equation by a_3 and obtain $x^3 + ax^2 + bx + c = 0$, the expression we will be returning to throughout our exploration.

In this part of the treatment of the cubic, we will solve the so-called depressed cubic in the way Scipione del Ferro did five hundred years ago (Nahin 8). The depressed cubic is one for which a is zero. In other words, it can be written as $x^3 + px = q$, where p and q are real non-zero. (If either of the two is zero, the solution to the equation requires no magic, so limitting the two variables is not symbolic of our ignorance.) Our assumption is slightly different from that of del Ferro, who assumed p and q to be positive, for the notion of negative numbers standing alone was unheard of at the time, even though subtraction was already standard procedure. Do not worry about this, however, for our generalisation is quite legal. Having defined our depressed cubic, we can express our unknown in terms of some u and v, namely as $x = u + v$. Rewriting the depressed cubic in these terms, we obtain $(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = 3uv(u+v) + (u^3+v^3)$ or $(u+v)^3 - 3uv(u+v) = u^3 + v^3$. Comparing terms in both versions of the cubic, we can see that $x = u + v$, $p = -3uv$, and $q = u^3 + v^3$. Solving the first for v, we get $v = \frac{p}{-3u}$, and solving the second for q by substituting for v, we obtain $q = u^3 - \frac{p^3}{27\,{}^3}$. This expression can be rewritten as $u^6 - qu^3 - \frac{p^3}{27} = 0$.

I hope that I have not brought about your ultimate panic over seeing such a monster. You might, indeed, be wondering, what the point is of transforming a cubic equation into a sextic one. Didn't they say that solving such an equation was beyond the realms of possible? Well, in the general case, yes. Fortunately, our monster happens to be a quadratic equation in disquise, and we may express its root u^3 as $u^3 = \frac{q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$. If we choose the positive root, solving for u, we get $u = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, and, since $v^3 = q - u^3$, $v = \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$. (Note that if we had turned our hand to the negative root, u would only swap expressions with v. Remember that our choice of u and v at the very beginning was… magical? Arbitrary. Let us maintain the facade of knowing what we are doing. Our choice was *deliberately* arbitrary.) Having obtained both u and v, we can substitute into our expression for x and get the somewhat scary-looking $x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$.

In other words, we have triumphantly found one root of our depressed cubic! From now on, if we apply all the knowledge we already have, we should be able to obtain the remaining two roots of the equation, for the fundamental theorem of algebra states that our depressed cubic can also be written in the form disclosing all of the cubic's roots, $x^3 + px - q = (x - x_1)(x - x_2)(x - x_3) = 0$. Hence, if we call the root we have just found x_1, all we need to do is to factor out $(x - x_1)$ by long division and solve the outcome as we would any other quadratic equation.

Now, it will have caught your attention that we have omitted a lot in our dealing with the cubic. First of all, we have only been investigating the depressed cubic, which may appear as a far cry from the general form. There is no need to worry about that, though, for it will prove to be astonishingly simple to solve this snag. We will make the transition in Part 2. However, there is one catch in how we have been working our way through all the substitution. We have made one assumption that makes the result elsewhere ambiguous, which we certainly want to avoid. I will leave you with that, for I think that you will find it quite pleasing to figure it out on your own. Notwithstanding, however you fare, I will do my best in Part 3 to reach a satisfactory conclusion of the conundrum in question. Are you intrigued? I hope so!

Sources

Nahin, Paul. *An Imaginary Tale: The Story of [the Square Root of Minus One]*. Princeton University Press, 1998.